

Comment on “*The Stochastic Nonlinear Schrödinger Equation in H^1* ”

Torquil Macdonald Sørensen

Centre of Mathematics for Applications
University of Oslo
NO-0316 Oslo, Norway
Email: t.m.sorensen@cma.uio.no, torquil@gmail.com

The paper “*The Stochastic Nonlinear Schrödinger Equation in H^1* ” [1] gives an existence proof for a stochastic nonlinear Schrödinger equation with multiplicative noise. We point out two mistakes that draw the validity of the proof into question.

Keywords (MSC2010): 35Q41 Time-dependent Schrödinger equations, Dirac equations; 35R60 Partial differential equations with randomness, stochastic partial differential equations; 35G20 Nonlinear higher-order equations.

1 Regarding the proof of [1, Theorem 4.1]

Consider [1, Theorem 4.1], concerning solution existence for the following stochastic nonlinear Schrödinger equation with multiplicative noise,

$$idu - (\Delta u + \lambda |u|^{2\sigma} u)dt = u dW - \frac{i}{2} u F_\phi dt, \quad (1.1)$$

describing a stochastic process u on \mathbb{R}^n . We refer to [1] for additional information about the mathematical details. Here we only describe the bare minimum of details that are necessary to describe our objection to the proof that is provided.

In the theorem, the following n -dependent parameter ranges for σ are assumed,

$$\begin{cases} 0 < \sigma & , n = 1, 2 \\ 0 < \sigma < 2 & , n = 3 \\ \frac{1}{2} \leq \sigma < \frac{2}{n-2} \quad \text{or} \quad \sigma < \frac{1}{n-1} & , n \geq 4. \end{cases} \quad (1.2)$$

The theorem essentially states that an *admissible pair* of Lebesgue space exponents (r, p) exists such that (1.1) has a unique solution in a certain function space characterised by (r, p) . Admissibility for (r, p) is defined as

$$r \geq 2, \quad \frac{2}{r} = n \left(\frac{1}{2} - \frac{1}{p} \right). \quad (1.3)$$

Dual Lebesgue space exponents are denoted by primed quantities, and are defined by the equation

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The proof given is for the special case $\sigma \geq 1/2$. In the proof, a second admissible pair (γ, s) is introduced after [1, Equation (4.17)]. The parameters s' and p are related through another parameter q which arises in the proof,

$$\frac{1}{s'} = \frac{2\sigma}{q} + \frac{1}{p}, \quad (1.4)$$

which is described prior to [1, Equation (4.18)]. The parameter q arises due to the use of the Sobolev embedding $H^1(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$. It is claimed that the embedding holds because “ $q < 2n/(n-3) < 2n/(n-2)$ ”. However, the second part of this inequality is incorrect, and therefore the Sobolev embedding is used without proper justification.

2 Regarding the proof of [1, Lemma 4.3]

In the proof of [1, Lemma 4.3], in the second estimate on page 121, the interpolation inequality for L^p -spaces, followed by the Sobolev embeddings $H^1(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ were used, where the parameter q was “as above”. Therefore, for the same reason, the Sobolev embeddings used here also lack a proper justification.

3 Conclusions

We believe that the errors described here put into question the validity of the existence proof provided in [1] for the SPDE in the multiplicative case.

References

- [1] A. de Bouard and A. Debussche, “The stochastic nonlinear Schrödinger equation in H^1 ,” *Stochastic Analysis and Applications* **21** no. 1, (2003) 97–126.